

Asymptotic Behavior of the Moments of the Maximum Queue Length During a Busy Period

Patrick Eschenfeldt
peschenfeldt@hmc.edu

Ben Gross
bgross@hmc.edu

Nicholas Pippenger
njp@math.hmc.edu

Department of Mathematics
Harvey Mudd College
1250 Dartmouth Avenue
Claremont, CA 91711

Abstract: We give a simple derivation of the distribution of the maximum L of the length of the queue during a busy period for the $M/M/1$ queue with $\lambda < 1$ the ratio between arrival rate and service rate. We observe that the asymptotic behavior of the moments of L is related to that of Lambert series for the generating functions for the sums of powers of divisors of positive integers. We show that $\text{Ex}[L] \sim \log(1/(1-\lambda))$ and $\text{Ex}[L^k] \sim k! \zeta(k)/(1-\lambda)^{k-1}$ for $k \geq 2$, so that $\text{Var}[L] \sim 3/\pi^2(1-\lambda)$. More generally, we show how to obtain asymptotic expansions for these moments with error terms of the form $O((1-\lambda)^N)$ for any N .

Keywords: Queueing theory, Lambert series, asymptotic expansions.

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1. Introduction

We study the $M/M/1$ queue (that is, the single server queue with independent exponentially distributed interarrival times and independent exponentially distributed service times). Let λ denote the ratio of the arrival rate (the reciprocal of the mean interarrival time) to the service rate (the reciprocal of the mean service time). When $\lambda < 1$, the busy period (the interval during which the service is continuously busy) is finite with probability one. Let the random variable L denote the maximum length of the queue during a busy period. (We count the customer being served in the queue length, so that $L \geq 1$ and $L = 1$ when and only when the busy period comprises a single service interval.) It is known that L has the distribution

$$\Pr[L > l] = \frac{(1 - \lambda) \lambda^l}{1 - \lambda^{l+1}}; \quad (1.1)$$

see for example Cohen [C1, pp. 191–193]. (Neuts [N] has also treated this question, but does not give a simple formula for the distribution.) In Section 2, we shall give a simple derivation of (1.1) based on the solution to the “Gambler’s Ruin” problem.

Our main goal in this paper is to give asymptotic expansions for the moments

$$\text{Ex}[L^k] = \sum_{l \geq 0} l^k \Pr[L = l]$$

of L in the “heavy traffic” limit as $\lambda \rightarrow 1$. Writing

$$\begin{aligned} \Delta_k(m) &= m^k - (m-1)^k \\ &= \sum_{0 \leq j \leq k-1} \binom{k}{j} (-1)^{k-1-j} m^j \end{aligned}$$

for the backward differences of the k -th powers, and setting

$$S_k(\lambda) = \sum_{m \geq 1} \frac{m^k \lambda^m}{1 - \lambda^m}, \quad (1.2)$$

summation by parts yields

$$\begin{aligned} \text{Ex}[L^k] &= \sum_{l \geq 0} l^k \Pr[L = l] \\ &= \sum_{l \geq 0} \Delta_k(l+1) \Pr[L > l] \\ &= (1 - \lambda) \sum_{l \geq 0} \frac{\Delta_k(l+1) \lambda^l}{1 - \lambda^{l+1}} \\ &= \frac{1 - \lambda}{\lambda} \sum_{m \geq 1} \frac{\Delta_k(m) \lambda^m}{1 - \lambda^m} \\ &= \frac{1 - \lambda}{\lambda} \sum_{m \geq 1} \sum_{0 \leq j \leq k-1} \binom{k}{j} (-1)^{k-1-j} \frac{m^j \lambda^m}{1 - \lambda^m} \\ &= \frac{1 - \lambda}{\lambda} \sum_{0 \leq j \leq k-1} \binom{k}{j} (-1)^{k-1-j} S_j(\lambda). \end{aligned} \quad (1.3)$$

Since $\text{Ex}[L^k]$ is a linear combination of the $S_k(\lambda)$, it will suffice to determine the asymptotic behavior of the sums $S_k(\lambda)$.

The sums in (1.2), which are called Lambert series, arise in a natural way in number theory (see for example Hardy and Wright [H, p. 257]). We have

$$\begin{aligned}
S_k(\lambda) &= \sum_{m \geq 1} \frac{m^k \lambda^m}{1 - \lambda^m} \\
&= \sum_{m \geq 1} m^k \sum_{l \geq 1} \lambda^{lm} \\
&= \sum_{l \geq 1} \sum_{m \geq 1} m^k \lambda^{lm} \\
&= \sum_{n \geq 1} \lambda^n \sum_{d|n} d^k \\
&= \sum_{n \geq 1} \sigma_k(n) \lambda^n,
\end{aligned} \tag{1.4}$$

where the inner sum in (1.4) is over integers d dividing n , and $\sigma_k(n)$ denotes the sum of the k -th powers of the divisors of n (see Hardy and Wright [H, p. 239]). Thus $S_k(\lambda)$ is the generating function for $\sigma_k(n)$.

We note that the Lambert series $S_k(\lambda)$ can be expressed in terms of known (albeit exotic) functions of analysis. We define the q -gamma function by

$$\Gamma_q(x) = (1 - q)^{1-x} \prod_{n \geq 0} \frac{1 - q^{n+1}}{1 - q^{n+x}}$$

(see for example Gasper and Rahman[G, p. 16]). (This function gets its name from the fact that $\lim_{q \rightarrow 1} \Gamma_q(x) = \Gamma(x)$, where $\Gamma(x)$ is the Euler gamma function; see for example Whittaker and Watson [W, pp. 235–264].) If we define the q -digamma function $\psi_q(x)$ as the logarithmic derivative

$$\begin{aligned}
\psi_q(x) &= \frac{\partial}{\partial x} \log \Gamma_q(x) \\
&= -\log(1 - q) + \log q \sum_{n \geq 0} \frac{q^{n+x}}{1 - q^{n+x}}
\end{aligned}$$

of the q -gamma function, then we have

$$S_0(\lambda) = \frac{\psi_\lambda(1) + \log(1 - \lambda)}{\log \lambda}.$$

To go further, we define the k -th q -polygamma function $\psi_q^{(k)}$ as the k -th derivative

$$\psi_q^{(k)}(x) = \left(\frac{\partial}{\partial x} \right)^k \psi_q(x)$$

of the q -digamma function. If we set $z = q^{n+x}$, then

$$\left(z \frac{\partial}{\partial z} \right) = \left(\frac{1}{\log q} \frac{\partial}{\partial x} \right).$$

Since

$$\sum_{m \geq 1} m^k z^m = \left(z \frac{\partial}{\partial z} \right)^k \frac{z}{1-z},$$

we have

$$\sum_{m \geq 1} m^k q^{(n+x)m} = \frac{1}{\log^k q} \left(\frac{\partial}{\partial x} \right)^k \frac{q^{n+x}}{1-q^{n+x}}.$$

Summing over $n \geq 0$ yields

$$\begin{aligned} \sum_{m \geq 1} \frac{m^k q^{xm}}{1-q^{xm}} &= \sum_{m \geq 1} m^k \sum_{n \geq 0} q^{m(n+x)} \\ &= \frac{1}{\log^k q} \left(\frac{\partial}{\partial x} \right)^k \sum_{n \geq 0} \frac{q^{n+x}}{1-q^{n+x}} \\ &= \frac{1}{\log^k q} \left(\frac{\partial}{\partial x} \right)^k \frac{\psi_q(x) + \log(1-q)}{\log q}. \end{aligned}$$

Thus for $k \geq 1$ we have

$$S_k(\lambda) = \frac{\psi_\lambda^{(k)}(1)}{\log^{k+1} \lambda}.$$

In Section 3, we shall begin our study of the asymptotic behavior of the moments of L , deriving the leading terms

$$\text{Ex}[L] \sim \log \frac{1}{1-\lambda}, \quad (1.5)$$

and, for $k \geq 2$,

$$\text{Ex}[L^k] \sim \frac{k! \zeta(k)}{(1-\lambda)^{-1}}, \quad (1.6)$$

where $\zeta(k) = \sum_{n \geq 1} 1/n^k$ is the Riemann zeta function. It will be noted that $\text{Ex}[L]$ grows quite slowly as $\lambda \rightarrow 1$. If the random variable K denotes the length of the queue in equilibrium, then $\text{Ex}[K] = \lambda/(1-\lambda)$, which grows much more rapidly (see Cohen [C], p. 181). It may appear paradoxical that the maximum queue length grows more slowly than the mean queue length, but it must be borne in mind that $\text{Ex}[L]$ is an average over busy periods, whereas $\text{Ex}[K]$ is an average over time. Indeed, the majority of busy periods have $L = 1$: after the arrival initiating the busy period, the next event determines whether $L = 1$ (if that event is a service termination) or $L > 1$ (if that event is another arrival). Because $\lambda < 1$, the former (with probability $1/(1-\lambda)$) is more likely than the latter (with probability $\lambda/(1-\lambda)$). We also note that, since $\zeta(2) = \pi^2/6$, we have $\text{Var}[L] = \text{Ex}[L^2] - \text{Ex}[L]^2 \sim \pi^2/3(1-\lambda)$, which grows much more rapidly than $\text{Ex}[L]$ (or even $\text{Ex}[L]^2$).

In Section 4, we shall refine these results by showing how to obtain complete asymptotic expansions for the moments $\text{Ex}[L^k]$ as $\lambda \rightarrow 1$. We obtain error terms of the form $O((1-\lambda)^c)$ for any c . These refinements are obtained by methods recently introduced in rigorous quantum field theory. All our results can be extended to the $M/M/s$ queue (with s independent and identical servers), if the busy period is defined as a contiguous interval during which all s servers are busy, and λ is the ratio of the arrival rate to s times the service rate for each server; for simplicity we confine our attention to the case $s = 1$.

2. The Distribution

In this section, we shall give a simple derivation of the formula (1.1). Consider a game played between two players: P , who begins with v dollars, and Q who begins with w dollars. At each step of the game, a biased coin is tossed; P wins with probability p , in which case Q pays P one dollar, and Q wins with the complementary probability $q = 1 - p$, in which case P pays Q one dollar. The game continues until one of the players is ruined (that is, has no money left). It is known that (1) with probability one, either P or Q is eventually ruined, and (2), if $p \neq q$, then the probability that Q is ruined is

$$\Pr[Q \text{ ruined}] = \frac{(q/p)^v - 1}{(q/p)^{v+w} - 1} \quad (2.1)$$

(see for example Feller [F, p. 345]).

Now consider a busy period of the $M/M/1$ queue. The successive events of arrivals and terminations of service intervals during the busy period correspond to steps in the game described above. The wealth of player P will correspond to the length of the queue at each step, so $v = 1$. An arrival will correspond to a win by player P , so $p = \lambda/(1 + \lambda)$, and the termination of a service interval will correspond to a win by player Q , so $q = 1/(1 + \lambda)$. Suppose that player Q begins with $w = l$ dollars. Then the event $L > l$ will correspond to Q being ruined. Substituting these values in (2.1) yields (1.1).

This correspondence also shows what happens for $\lambda \geq 1$. For $\lambda = 1$ (in which case the busy period is finite with probability one, but its expected length is infinite), we have take $p = q = 1/2$, and have

$$\Pr[Q \text{ ruined}] = \frac{v}{v + w}.$$

This result yields

$$\Pr[L > l] = \frac{1}{l + 1},$$

so that

$$\text{Ex}[L] = \sum_{l \geq 0} \Pr[L > l] \quad (2.2)$$

diverges logarithmically. Of course, for $\lambda > 1$ (in which case the busy period is infinite with positive probability), (2.1) shows that (2.2) diverges linearly.

3. The Leading Terms of the Moments

In this section, we shall derive the leading terms (1.5) and (1.6) of the asymptotic expansions for the moments of L . We begin by deriving the leading terms of the asymptotic expansions for the sums

$$S_0(\lambda) \sim \frac{1}{1 - \lambda} \log \frac{1}{1 - \lambda} \quad (3.1)$$

and, for $j \geq 1$,

$$S_j(\lambda) \sim \frac{j! \zeta(j + 1)}{(1 - \lambda)^{j+1}}. \quad (3.2)$$

Once these formulas are established, it will be clear that the sum in (1.3) is dominated by the term for which $j = k - 1$, so that $\text{Ex}[L^k] \sim k S_{k-1}(\lambda)$, and (1.5) and (1.6) follow from (3.1) and (3.2), respectively.

Our strategy for proving (3.1) and (3.2) will be to approximate the sums $S_j(\lambda)$ by integrals

$$I_j(\lambda) = \int_1^\infty \frac{x^j \lambda^x dx}{1 - \lambda^x},$$

then then to show that the difference $S_j(\lambda) - I_j(\lambda)$ is negligible in comparison with $I_j(\lambda)$. It will be convenient to write $\lambda = e^{-h}$. The limit $\lambda \rightarrow 1$ then corresponds to $h \rightarrow 0$. We have

$$\begin{aligned} h &= \log \frac{1}{\lambda} \\ &= \log \frac{1}{1 - (1 - \lambda)} \\ &\sim 1 - \lambda. \end{aligned} \tag{3.3}$$

For $j = 0$, we have

$$\begin{aligned} I_0(\lambda) &= \int_1^\infty \frac{\lambda^x dx}{1 - \lambda^x} \\ &= \int_1^\infty \sum_{l \geq 1} \lambda^{lx} dx \\ &= \sum_{l \geq 1} \int_1^\infty e^{-h l x} dx \\ &= \sum_{l \geq 1} \frac{e^{-h l}}{h l} \\ &= \frac{1}{h} \sum_{l \geq 1} \frac{\lambda^l}{l} \\ &= \frac{1}{h} \log \frac{1}{1 - \lambda}. \end{aligned}$$

Substituting (3.3) in this result yields

$$I_0(\lambda) \sim \frac{1}{1 - \lambda} \log \frac{1}{1 - \lambda}. \tag{3.4}$$

We bound $|S_0(\lambda) - I_0(\lambda)|$ by the total variation of $f(x) = \lambda^x/(1 - \lambda^x)$. Since $f(x)$ decreases monotonically from $\lambda/(1 - \lambda)$ to 0 as x increases from 1 to ∞ , we have $|S_0(\lambda) - I_0(\lambda)| \leq \lambda/(1 - \lambda) \sim 1/(1 - \lambda)$. Since this difference is negligible in comparison with (3.4), we obtain (3.1).

For $j \geq 1$, we have

$$\begin{aligned} I_j(\lambda) &= \int_1^\infty \frac{x^j \lambda^x dx}{1 - \lambda^x} \\ &= \int_0^\infty \frac{(y+1)^j \lambda^{y+1} dy}{1 - \lambda^{y+1}} \\ &= \int_0^\infty \sum_{0 \leq i \leq j} \binom{j}{i} \frac{y^i \lambda^{y+1} dy}{1 - \lambda^{y+1}} \\ &= \int_0^\infty \sum_{0 \leq i \leq j} \binom{j}{i} y^i \sum_{l \geq 1} \lambda^{l(y+1)} dy \\ &= \int_0^\infty \sum_{0 \leq i \leq j} \binom{j}{i} y^i \sum_{l \geq 1} e^{-h l (y+1)} dy \end{aligned}$$

$$\begin{aligned}
&= \sum_{0 \leq i \leq j} \binom{j}{i} \sum_{l \geq 1} \int_0^\infty y^i e^{-hl(y+1)} dy \\
&= \sum_{0 \leq i \leq j} \binom{j}{i} \sum_{l \geq 1} \frac{i! e^{-hl}}{(hl)^{i+1}} \\
&= \sum_{0 \leq i \leq j} \binom{j}{i} \frac{i!}{h^{i+1}} \sum_{l \geq 1} \frac{\lambda^l}{l^{i+1}} \\
&= \sum_{0 \leq i \leq j} \binom{j}{i} \frac{i!}{h^{i+1}} \text{Li}_{i+1}(\lambda),
\end{aligned} \tag{3.5}$$

where $\text{Li}_k(\lambda) = \sum_{n \geq 1} \lambda^n / n^k$ is the k -th polylogarithm. Since $\text{Li}_1(\lambda) = \log(1/(1-\lambda))$ and $\text{Li}_k(\lambda) \rightarrow \zeta(k)$ as $\lambda \rightarrow 1$ for $k \geq 2$, the sum in (3.5) is dominated by the term for which $i = j$, and we have

$$\begin{aligned}
I_j(\lambda) &\sim \frac{j! \zeta(j+1)}{h^{j+1}} \\
&\sim \frac{j! \zeta(j+1)}{(1-\lambda)^{j+1}}
\end{aligned} \tag{3.6}$$

We bound $|S_j(\lambda) - I_j(\lambda)|$ by the total variation of $f(x) = x^j \lambda^x / (1 - \lambda^x)$ for $0 \leq x < \infty$. As x increases, $f(x)$ increases monotonically from 0 to a maximum, then decreases monotonically to 0. Thus the total variation of $f(x)$ is twice the maximum. This maximum is

$$\begin{aligned}
\max_{0 \leq x < \infty} f(x) &= \max_{0 \leq x < \infty} \frac{x^j e^{-hx}}{1 - e^{-hx}} \\
&= \max_{0 \leq x < \infty} \frac{x^j}{e^{hx} - 1} \\
&= \frac{1}{hj} \max_{0 \leq y < \infty} \frac{y^j}{e^y - 1}.
\end{aligned}$$

Furthermore, $y^j / (e^y - 1) \leq j!$, because $e^y - 1 = \sum_{n \geq 1} y^n / n! \geq y^j / j!$. Thus $|S_j(\lambda) - I_j(\lambda)| \leq 2 \max_{0 \leq x < \infty} f(x) \leq 2j! / h^j \sim 2j! / (1 - \lambda)^j$. Since this difference is negligible in comparison with (3.6), we obtain (3.2).

4. The Complete Asymptotic Expansions

In this section we shall show how asymptotic expansions, with error terms of the form $O((1-\lambda)^N)$ for any N , can be derived for all of the moments $\text{Ex}[L^k]$. The essence of the argument is to use the Euler-Maclaurin formula to estimate the difference between $S_j(\lambda)$ and $I_j(\lambda)$. This is most conveniently done using a result of Zagier [Z]. Indeed, for $j \geq 1$, Zagier gives the expansion for $S_j(\lambda)$, in terms of the parameter $h = -\log \lambda$ rather than $1 - \lambda$. All that remains for us to do is substitute an expansion for h in terms of $1 - \lambda$. For $j = 0$, the expansion for $S_0(\lambda)$ in terms of h has been given by Egger (né Endres) and Steiner [E1, E2], again using the result of Zagier. We shall proceed differently, to obtain an expansion involving $-\log(1 - \lambda)$ rather than $-\log h$.

Proposition: (Zagier [Z, p. 318]) Let $f(x)$ be analytic at $x = 0$, with power series $f(x) = \sum_{n \geq 0} b_n x^n$ about $x = 0$. Suppose that $\int_0^\infty |f^{(N)}(x)| dx < \infty$ for all $N \geq 0$, where $f^{(N)}(x)$ denotes the N -th derivative of $f(x)$. Define $I_f = \int_0^\infty f(x) dx$. Let $g(x) = \sum_{m \geq 1} f(mx)$. Then $g(x)$ has the asymptotic expansion

$$g(x) \sim \frac{I_f}{x} + \sum_{n \geq 0} \frac{b_n B_{n+1} (-1)^n x^n}{(n+1)}, \tag{4.1}$$

where B_k is the k -th Bernoulli number, defined by $t/(e^t - 1) = \sum_{k \geq 0} B_k t^k/k!$.

This result is proved by using the Euler-Maclauren formula,

$$\begin{aligned} \int_0^M f(y) dy &= \frac{f(0)}{2} + \sum_{1 \leq m \leq M-1} f(m) + \frac{f(M)}{2} + \sum_{1 \leq n \leq N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} (f^{(n)}(M) - f^{(n)}(0)) \\ &\quad + (-1)^N \int_0^M f^{(N)}(y) \frac{B_N(\{y\})}{N!} dy, \end{aligned}$$

where $B_k(y)$ is the k -th Bernoulli polynomial, defined by $te^{yt}/(e^t - 1) = \sum_{k \geq 0} B_k(y) t^k/k!$, and $\{y\} = y - \lfloor y \rfloor$ denotes the fractional part of y . (For the Euler-Maclauren formula, the Bernoulli numbers and the Bernoulli polynomials, see for example Whittaker and Watson [W, pp. 125–128], where, however, the indexing of the numbers and polynomials is different.) The condition $\int_0^\infty |f^{(N)}(y)| dy < \infty$ allows us to let $M \rightarrow \infty$, obtaining

$$\int_0^\infty f(y) dy = \sum_{m \geq 1} f(m) + \sum_{0 \leq n \leq N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0) + (-1)^N \int_0^\infty f^{(N)}(y) \frac{B_N(\{y\})}{N!} dy.$$

If we now write $f(xy)$ instead of $f(y)$, we obtain

$$\int_0^\infty f(xy) dy = \sum_{m \geq 1} f(mx) + \sum_{0 \leq n \leq N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0) x^n + (-1)^N x^N \int_0^\infty f^{(N)}(xy) \frac{B_N(\{y\})}{N!} dy.$$

Changing the variable of integration from y to y/x then yields

$$\frac{1}{x} \int_0^\infty f(y) dy = \sum_{m \geq 1} f(mx) + \sum_{0 \leq n \leq N-1} \frac{(-1)^n B_{n+1}}{(n+1)!} f^{(n)}(0) x^n + (-1)^N x^{N-1} \int_0^\infty f^{(N)}(y) \frac{B_N(\{y/x\})}{N!} dy.$$

The integral on the left-hand side is I_f , the first sum on the right-hand side is $g(x)$, $f^{(n)}(0) = n! b_n$, and the last term on the right-hand side is $O(x^{N-1})$. Thus

$$\frac{I_f}{x} = g(x) + \sum_{0 \leq n \leq N-1} \frac{b_n B_{n+1} (-1)^n x^n}{(n+1)!} + O(x^{N-1}),$$

which yields the expansion (4.1).

For $j \geq 1$, we define

$$f(x) = \frac{x^j}{e^x - 1}.$$

Then $f(x)$ is analytic at $x = 0$ with the Taylor series

$$f(x) = \sum_{n \geq 0} \frac{B_n x^{n+j-1}}{n!}$$

and the integral

$$\begin{aligned} I_f &= \int_0^\infty \frac{x^j e^{-x} dx}{1 - e^{-x}} \\ &= j! \zeta(j+1) \end{aligned}$$

(see for example Whittaker and Watson [W, p. 266]). Furthermore, $f^{(N)}(x)$ is a rational function of x and e^x , in which the degree of the numerator in e^x is N , while the denominator is $(e^x - 1)^{N+1}$. Thus $f(x)$ satisfies the conditions of the proposition, and we have the asymptotic expansion

$$g(x) \sim \frac{j! \zeta(j+1)}{x} + \sum_{n \geq 0} \frac{(-1)^{n+j-1} B_n B_{n+j} x^{n+j-1}}{n! (n+j)}.$$

Recalling that $\lambda = e^{-h}$, so that $h = -\log \lambda$, we therefore have

$$\begin{aligned} S_j(\lambda) &= \sum_{m \geq 1} \frac{m^j e^{-hm}}{1 - e^{-hm}} \\ &= \frac{1}{h^j} \sum_{m \geq 1} \frac{(mh)^j}{e^{hm} - 1} \\ &= \frac{1}{h^j} \sum_{m \geq 1} f(mh) \\ &= \frac{g(h)}{h^j} \\ &\sim \frac{j! \zeta(j+1)}{h^{j+1}} + \sum_{n \geq 0} \frac{(-1)^{n+j-1} B_n B_{n+j} h^{n-1}}{n! (n+j)}. \end{aligned} \tag{4.2}$$

We note that, if j is odd, then this expansion has only finitely many terms (because $B_n = 0$ for odd $n \geq 3$). To obtain an asymptotic expansion in terms of $1 - \lambda$, we must substitute the expansion for $1/h$:

$$\begin{aligned} \frac{1}{h} &= \frac{1}{-\log \lambda} \\ &= \frac{-1}{\log(1 - (1 - \lambda))} \\ &= \frac{1}{1 - \lambda} \frac{-(1 - \lambda)}{\log(1 - (1 - \lambda))} \\ &= \frac{1}{1 - \lambda} \sum_{n \geq 0} \frac{(-1)^n C_n (1 - \lambda)^n}{n!}, \end{aligned} \tag{4.3}$$

where C_k is the k -th Bernoulli number of the second kind, defined by $t/\log(1+t) = \sum_{k \geq 0} C_k t^k/k!$ (see for example Roman [R, p. 116]). (These numbers are also called the Cauchy numbers of the first kind, and are given by $C_k = \int_0^1 x(x-1) \cdots (x-k+1) dx$; see for example Comtet [C2, pp. 293–294].)

For $j = 0$, we must proceed differently, because

$$f(x) = \frac{1}{e^x - 1}$$

has a pole at $x = 0$. We define

$$\begin{aligned} f^*(x) &= f(x) - \frac{e^{-x}}{x} \\ &= \frac{1}{e^x - 1} - \frac{e^{-x}}{x}. \end{aligned}$$

Then $f^*(x)$ is analytic at $x = 0$ with the Taylor series

$$f^*(x) = \sum_{n \geq 0} \frac{(B_{n+1} - (-1)^{n+1}) x^n}{(n+1)!}$$

and the integral

$$\begin{aligned} I_{f^*} &= \int_0^\infty \left(\frac{1}{e^x - 1} - \frac{e^{-x}}{x} \right) dx \\ &= \gamma \end{aligned}$$

(see for example Whittaker and Watson [W, p. 246]). Furthermore, $f^{*(N)}(x)$ is a rational function of x and e^x , in which the degree of the numerator in e^x is N , while the denominator is $((e^x - 1)x)^{N+1}$. Thus $f^*(x)$ satisfies the conditions of the proposition, and we have the asymptotic expansion

$$g^*(x) \sim \frac{\gamma}{x} + \sum_{n \geq 0} \frac{(-1)^n B_{n+1} (B_{n+1} - (-1)^{n+1}) x^n}{(n+1)(n+1)!}.$$

We therefore have

$$\begin{aligned} S_0(\lambda) &= \sum_{m \geq 1} \frac{e^{-mh}}{1 - e^{-mh}} \\ &= \sum_{m \geq 1} \frac{1}{e^{mh} - 1} \\ &= \sum_{m \geq 1} \frac{e^{-mh}}{mh} + \sum_{m \geq 1} \frac{1}{e^{mh} - 1} - \frac{e^{-mh}}{mh} \\ &= \frac{1}{h} \log \frac{1}{1 - \lambda} + \sum_{m \geq 1} f^*(mh) \\ &= \frac{1}{h} \log \frac{1}{1 - \lambda} + g^*(h) \\ &\sim \frac{1}{h} \log \frac{1}{1 - \lambda} + \frac{\gamma}{h} + \sum_{n \geq 0} \frac{(-1)^n B_{n+1} (B_{n+1} - (-1)^{n+1}) h^n}{(n+1)(n+1)!}. \end{aligned} \tag{4.4}$$

To obtain asymptotic expansions for the moments of L , we substitute (4.3) into (4.2) and (4.4), then substitute the results into (1.3), using the expansion

$$\begin{aligned} \frac{1 - \lambda}{\lambda} &= \frac{1 - \lambda}{1 - (1 - \lambda)} \\ &= \sum_{n \geq 1} (1 - \lambda)^n. \end{aligned}$$

Retaining only terms that do not vanish as $\lambda \rightarrow 1$, we obtain

$$\text{Ex}[L] = \log \frac{1}{1 - \lambda} + \gamma + O\left((1 - \lambda) \log \frac{1}{1 - \lambda}\right)$$

and

$$\text{Ex}[L^2] = \frac{3\pi^2}{6(1 - \lambda)} + \log \frac{1}{1 - \lambda} + (\gamma - 1) + O\left((1 - \lambda) \log \frac{1}{1 - \lambda}\right)$$

for the first two moments. Thus we have

$$\begin{aligned} \text{Var}[L] &= \text{Ex}[L^2] - \text{Ex}[L]^2 \\ &= \frac{3\pi^2}{6(1 - \lambda)} - \log^2 \frac{1}{1 - \lambda} + (1 - 2\gamma) \log \frac{1}{1 - \lambda} - \gamma^2 + O\left((1 - \lambda) \log^2 \frac{1}{1 - \lambda}\right). \end{aligned}$$

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6. References

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